OBLIQUE INCIDENCE OF AN ELECTROMAGNETIC WAVE ON A PARABOLIC PLASMA LAYER

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We consider the oblique incidence of a sinusoidal electromagnetic wave (with the electric vector in the plane of incidence) on a parabolic layer of a plane-layered isotropic plasma. The reflection and transmission factors are obtained, and also the electromagnetic-plasma wave conversion factor.

1. Consider the oblique incidence of a sinusoidal wave, with the electric vector in the plane of incidence, on a parabolic layer of a plane-layered isotropic plasma. The electric and magnetic waves in the plasma, in this case, are described by the following equations [1]:

$$H_x = G(z)\, e^{i(\omega t - \Omega\alpha_0 y)}, \quad E_y = -\frac{i}{\Omega\varepsilon'}\frac{\partial H_x}{\partial z}, \quad E_z = \frac{i}{\Omega\varepsilon'}\frac{\partial H_x}{\partial y} \tag{1.1}$$

$$\frac{d^2G}{dz^2} - \frac{1}{\varepsilon'}\frac{d\varepsilon'}{dz}\frac{dG}{dz} + \Omega^2(\varepsilon' - \alpha_0^2)G = 0 \quad \left(\Omega = \frac{\omega}{c}, \quad \omega = 2\pi f\right) \tag{1.2}$$

$$\varepsilon' = 1 - \frac{f_*^2}{f^2}\left(1 - \frac{z^2}{z_m^2}\right) - is, \quad \alpha_0 = \sin\theta_0, \quad i = \sqrt{-1} \tag{1.3}$$

Here f is the frequency, c is the speed of light, θ_0 is the angle of incidence of the wave on the layer, ε' is the complex dielectric constant, f_* is the critical frequency, and z_m is the half-thickness of the layer. We assume that the absorption s is constant along the layer. The case of a linear layer has been considered in detail in [2]. The case of a second-order zero ($s = 0, f = f_*$, $\varepsilon' = \varepsilon = z^2/z_m^2$) was partly considered in [3]; however, the field was considered mainly in the region $\varepsilon' = 0$. The reflection and transmission factors were not calculated explicitly and were not analyzed. Since the wave equation can be solved accurately for $\varepsilon' = z^2/z_m^2$, it is of special interest to obtain these factors. We therefore consider them first.

2. Consider the particular case

$$\varepsilon' = \varepsilon = z^2 / z_m^2, \quad s = 0, \quad f = f_*$$

With these assumptions Eq. (1.2) becomes

$$\frac{d^2G}{dz^2} + \frac{Q}{z}\frac{dG}{dz} + (a + bz^2)G = 0 \tag{2.1}$$

In this case the arbitrary parameters Q, a, and b take the values

$$Q = -2, \quad a = -\Omega^2\alpha_0^2, \quad b = \omega_*^2 / c^2 z_m^2, \quad \omega_* = 2\pi f_*$$

The $G^{(1)}$ and $G^{(2)}$ solutions of this equation can be expressed in terms of Whittaker functions [4]

$$G^{(1)}(i\tau) = e^{-i\pi\eta}(i)^{\eta}\tau^{1/4}W_{-\eta,\mu}(i\tau e^{-i\pi}) \sim \tau^{1/2 r_1}e^{1/2 i\tau}(1 + O(\tau^{-1})) \quad (|\tau| \to \infty)$$
$$G^{(2)}(i\tau) = (i)^{-\eta}\tau^{1/4}W_{\eta,\mu}(i\tau) \sim \tau^{1/2 r_2}e^{-1/2 i\tau}(1 + O(\tau^{-1})) \quad (|\tau| \to \infty)$$
$$\tau = z^2 b^{1/2} = 2\pi z^2/\lambda_* z_m, \quad \lambda_* = c / f_*$$
$$\eta = {}^1/_4(r_2 - r_1), \quad \mu = {}^1/_4(\rho_1 - \rho_2), \quad r_{1,2} = {}^1/_2(1 \pm iab^{-1/2}) \tag{2.2}$$

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Here ρ_1 and ρ_2 represent the behavior of the solutions in the neighborhood of the ordinary singular point of the equation z = 0, and can be obtained from the defining equation

$$\rho(\rho - 1) + \rho Q = 0 \tag{2.3}$$

If the incident wave propagates from the side z < 0, then $G^{(2)}$ describes a transmitted wave for z > 0. Correspondingly, for z < 0, $G^{(2)}$ describes a reflected wave and $G^{(1)}$ describes an incident wave. Starting from Eqs. 9.120, 9.231 (2), and 9.232 of [4], we can obtain

$$W_{\eta,\mu}(ze^{\pm i2\pi}) = \frac{2\pi i e^{-i\pi\eta} W_{-\eta,\mu}(ze^{-i\pi})}{\Gamma(1/2 + \mu - \eta)\Gamma(1/2 - \mu - \eta)} - (q_{\pm} 2\cos 2\pi\mu + e^{-i2\pi\eta}) W_{\eta,\mu}(z) \tag{2.4}$$

$$q_+ = 1, \qquad q_- = 0 \quad (\Gamma \text{ — is a gamma function})$$

(A formula for $W_{\eta,\mu}(ze^{i2\pi})$, is given in [3], but due to a misprint $e^{i2\pi\eta}$ is written for the second term.)

From (2.2) and (2.4) the following relationship between the solutions $G^{(1)}$ and $G^{(2)}$ is obtained on the positive and negative semiaxes z:

$$G^{(2)}(i\tau e^{\pm i2\pi}) = -\frac{2\pi (i)^{-2\eta} G^{(1)}(i\tau)}{\Gamma(1/2 + \mu - \eta)\Gamma(1/2 - \mu - \eta)} - i(q_{\pm} 2\cos 2\pi\mu + e^{-i2\pi\eta}) G^{(2)}(i\tau) \tag{2.5}$$

Equation (2.5), in particular, gives the relationship between the asymptotic solutions of Eq. (2.1) on the positive and negative semiaxes z, as $|\tau| \to \infty$. As can be seen from (2.5), the nature of this relation ship depends on the direction in which point z is circumvented (over the upper (q_+) or lower (q_-) half-planes of the complex z-plane). This means that the solutions in z = 0, generally speaking, have a branch point. Hence, in general, the problem arises as to the correct choice of the direction in which one circumvents the point (see section (4.5). The following expressions are obtained from (2.5) for the amplitude reflection factor R (the ratio of the reflected wave amplitude to the incident wave amplitude) and for the transmission factor D:

$$D = (2\pi)^{-1} e^{i\pi(\eta-1)} \Gamma(1/2 + \mu - \eta)\Gamma(1/2 - \mu - \eta)$$
$$R = e^{i3/2\pi}(q_{\pm} 2\cos 2\pi\mu + e^{-i2\pi\eta}) D \tag{2.6}$$

In the case considered Q = −2, $\mu = 3/4$, $\cos 2\pi\mu = 0$, i.e., the directions of rotation are equivalent. Substituting the actual expressions for Q, a, and b in (2.6), and also using the properties of gamma functions [see [4], Eq. 8.331, 8.335 (1), and 8.332 (2)], we can obtain the following expressions for the reflection factor $|R|^2$ (the ratio of the reflected wave intensity to the incident wave intensity) and the transmission factor $|D|^2$:

$$|R|^2 = \frac{e^h}{1 + e^h}, \qquad |D|^2 = \frac{1}{1 + e^h}, \qquad h = 2\pi^2 \frac{z_m}{\lambda_*} \alpha_0^2 \tag{2.7}$$

Hence we see that for s = 0 and $f = f_*$ there is no absorption in the region of the poles ($|R|^2 + |D|^2 = 1$), and the presence of an infinite electric vector amplitude ($|E_z| \sim 1/z^2, |E_y| \sim 1/z$ as $z \to 0$) must be regarded as the result of a transition process [1]. Putting $G = w\varepsilon^{1/2}$ Eq. (2.1) takes the form

$$\frac{d^2 w}{dz^2} + q^2 w = 0, \qquad q^2 = -\frac{2}{z^2} + \frac{\omega_*^2}{c^2}\left(\frac{z^2}{z_m^2} - \alpha_0^2\right)$$

Figure 1 shows q^2 as a function of z. In the above calculation we have ignored reflection from the jump $d\varepsilon/dz$ at the points $z = \pm z_m$. This is valid when $(z_m - z_0)/\lambda_* \gg 1$ (Fig. 1) and the transition to $\varepsilon = 1$ can be smoothed out in the same way. This condition formally limits the applicability of (2.7) to large θ_0. However, in practice, total reflection from the layer begins at values of θ_0 for which (2.7) holds. If, as an example, for $z_m/\lambda_* = 10$ it is required that $(z_m - z_0)/z_m \geq 1/2$, the condition $\theta_0 \leq 30°$ ($h \leq 48.8$) must be satisfied. In fact, total reflection occurs for $\theta_0 < 30°$.

3. Let us consider the relationship between the asymptotic solutions in the general case. For a parabolic layer Eq. (1.2) has the form

$$\frac{d^2 G}{dz^2} - \left(\frac{1}{z + z_1} + \frac{1}{z - z_1}\right)\frac{dG}{dz} + \Omega^2 [u^2 (z^2 - z_1^2) - \alpha_0^2] G = 0$$

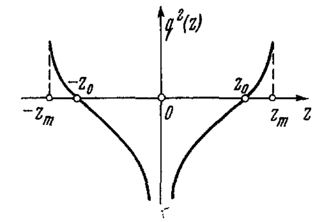

Fig. 1

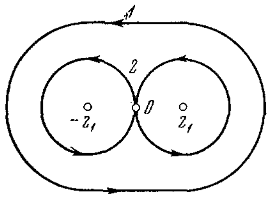

Fig. 2

$$z_1^2 = z_m^2\left[1-\frac{f^2}{f_*^2}(1-is)\right], \quad u^2=\frac{f_*^2}{f^2 z_m^2} \tag{3.1}$$

In more general form this equation is of the following type:

$$\frac{d^2G}{dz^2}+\frac{Qz}{z^2-z_1^2}\frac{dG}{dz}+(a+bz^2)G=0 \tag{3.2}$$

where Q, a, and b are arbitrary parameters. When $z_1 = 0$ it reduces to Eq. (2.1), which is a particular case of (3.2). Let us compare these two equations.

Equation (2.1) does not change when z is replaced by $ze^{i\pi}$. It has two singular points: an intrinsically singular point at $z = \infty$ and an ordinary singular point at $z = 0$. Its asymptotic solutions [see (2.2)] for large $|\tau|$ for $z > 0$ and $z < 0$ are related by expression (2.5). As seen from (2.5), the factors in the asymptotic solutions depend only on the parameters $\rho_{1,2}$ and $r_{1,2}$, which characterize the singular points [$\mu = {}^1\!/_4(\rho_1-\rho_2)$; $\eta = {}^1\!/_4(r_2-r_1)$]. The effect of the pole at $z = 0$ (and consequently, of the singular point of the solution) on this relationship occurs only through the variation in μ. Thus, if we put $Q = 0$ in (2.1), i.e., we consider the equation without the pole, we must put $\mu = {}^1\!/_4$ in (2.5), since in this case $\rho_1 = 1$, $\rho_2 = 0$. Here $r_{1,2}$ and η do not vary.

Equation (3.2) does not change when z is replaced by $ze^{i\pi}$. It also has an intrinsic singular point at $z = \infty$, and its asymptotic solutions for large $|\tau|$ ($|z| > |z_1|$) have the same principal terms as for Eq. (2.1), i.e., $\tau^{r_{1,2}/2}e^{\pm i\tau/2}$, and are represented by the same $r_{1,2} = {}^1\!/_2(1 \pm iab^{-1/2})$ [5]. It only differs from (2.1) in that it has two simple poles at the points $z = \pm z_1$, which merge into a single simple pole as $z_1 \to 0$, or, in other words, two ordinary singular points $z = \pm z_1$, which merge into one, likewise ordinary, singular point as $z_1 \to 0$. Equations (2.1) and (3.2) are essentially of the same type.

It is therefore natural to suggest that for Eq. (3.2) the connection between the asymptotic solutions is given by the same expression (2.5) as for Eq. (2.1), but with μ replaced by μ', where the parameter μ' must represent two normal singular points ($z = \pm z_1$) in the same way that each μ characterizes one ordinary singular point ($z = 0$). It is obvious that μ' depends on z_1 and $\mu' = \mu$ when $z_1 = 0$. Correspondingly, in the general case the amplitude reflection factor R and the amplitude transmission factor D are described by (2.6) with μ replaced by μ'. Hence, the problem reduces to determining μ'. Note that the condition $|z| > |z_1|$ reduces to the condition $|z_1| < z_m$, which imposes limitations on the region of frequencies considered.

Setting $G = w(z^2-z_1^2)^{1/2}$, Eq. (3.1) reduces to the form

$$\frac{d^2w}{dz^2}+\left\{-\frac{3}{4}\frac{1}{(z-z_1)^2}-\frac{1}{4z_1(z-z_1)}-\frac{3}{4}\frac{1}{(z+z_1)^2}+\frac{1}{4z_1(z+z_1)}+\Omega^2\left[u^2(z^2-z_1^2)-\alpha_0^2\right]\right\}w=0 \tag{3.3}$$

Its solutions in the neighborhood of the points $z = \pm z_1$ are

$$\begin{gathered}
y_1=(z-z_1)^{\rho_1}\sum_{k=0}^{\infty}C_k(z_1)(z-z_1)^k, \qquad x_1=(z+z_1)^{\rho_1}\sum_{k=0}^{\infty}(-1)^kC_k(z_1)(z+z_1)^k \\
y_2=ry_1\ln(z-z_1)+(z-z_1)^{\rho_2}\sum_{k=0}^{\infty}d_k(z_1)(z-z_1)^k \\
x_2=rx_1\ln(z+z_1)+(z+z_1)^{\rho_2}\sum_{k=0}^{\infty}(-1)^kd_k(z_1)(z+z_1)^k \\
\rho_1={}^3\!/_2,\quad \rho_2=-{}^1\!/_2,\quad C_0=1,\quad d_0=-{}^1\!/_2,\quad r=-\Omega^2\,\alpha_0^2/4
\end{gathered} \tag{3.4}$$

where r is determined by the Frobenius method. As can easily be shown from recurrence relations, $C_k(z_1)$ is an even function of z_1 if k is even, and is an odd function if k is odd. To derive μ' it is convenient to use matrix notation. We set

$$Y=\begin{Vmatrix} y_1 \\ y_2 \end{Vmatrix}, \qquad X=\begin{Vmatrix} x_1 \\ x_2 \end{Vmatrix} \tag{3.5}$$

The linearly independent solutions Y after going around both singular points along contour 1 (see Fig. 2) reduce to the solution Y", where

$$Y'' = CY, \qquad C = \| c_{jk} \|$$

where C is a nonsingular constant matrix. In principle, we can choose a pair of solutions

$$V = \left\| \begin{matrix} v_1 \\ v_2 \end{matrix} \right\|, \qquad V = BY$$

(B is a constant nonsingular matrix) such that after going around the singularities along contour 1 either

$$v_1'' = \xi_1' v_1, \; v_2'' = \xi_2' v_2$$

(if C is similar to a diagonal matrix) or

$$v_1'' = \xi_1' v_1, \quad v_2'' = c_{21} v_1 + \xi'_1 v_2$$

(if C is not similar to a diagonal matrix). Here ξ_1', and ξ_2' are characteristic numbers of matrix C, i.e., they can be obtained from the equation

$$\begin{vmatrix} c_{11} - \xi' & c_{12} \\ c_{21} & c_{22} - \xi' \end{vmatrix} = 0 \tag{3.6}$$

The matrix C depends on the choice of the pair of solutions, but ξ_1', and ξ_2' do not. The numbers ξ_1', and ξ_2' are determined only by the nature of the singular points. If a similar construction is carried out for one singular point and appropriate characteristic numbers ξ_1 and ξ_2 are obtained, then, as is well known [6],

$$\rho_1 = \frac{\ln \xi_1}{2\pi i} + n_1, \quad \rho_2 = \frac{\ln \xi_2}{2\pi i} + n_2, \quad \mu = \frac{1}{4}\left(\frac{1}{2\pi i} \ln \frac{\xi_1}{\xi_2} + (n_1 - n_2)\right)$$

where n_1 and n_2 are certain integers. Hence for two singular points

$$\mu' = \frac{1}{4}\left(\frac{1}{2\pi i} \ln \frac{\xi_1'}{\xi_2'} + l\right) \tag{3.7}$$

The value of the integer l must be chosen from the condition $\mu'(z_1) = \mu$ with $z_1 = 0$ (see below). To obtain C we note that

$$\begin{gathered} Y' \equiv Y\left((z - z_1)e^{i2\pi}\right) = TY(z - z_1) \\ X' \equiv X\left((z + z_1)e^{i2\pi}\right) = TX(z + z_1) \\ Y = AX, \quad T = \| t_{jk} \|, \quad t_{11} = t_{22} = e^{i2\pi\rho_1}, \quad t_{21} = i2\pi r e^{i2\pi\rho_1}, \; t_{12} = 0 \end{gathered} \tag{3.8}$$

while the elements α_{jk} of matrix A (which relates the solution X to the solutions Y) can be found from the following equations:

$$Y_{z=0} = AX_{z=0}, \quad \frac{dY}{dz}_{z=0} = A\frac{dX}{dz}_{z=0} \tag{3.9}$$

Going around both singular points along the path 2 (Fig. 2) is equivalent to going around along path 1. Beginning at z = 0, we go around $z = z_1$ with the solutions Y along the right loop of path 2. This gives Y' = TY = TAX. By further going around $z = -z_1$ with the solutions Y' along the left loop of the path and using (3.8), we have

$$C = TATA^{-1} \tag{3.10}$$

Substituting (3.10) into (3.6) gives

$$\xi'^2 - \left(2 + \frac{\alpha_{12}^2 4\pi^2 r^2}{\det A}\right)\xi' + 1 = 0 \tag{3.11}$$

Finally, using the symmetry of the coefficients in Y and X [see (3.4)], we have

$$\mu' = \frac{1}{8\pi i} \ln \varkappa + \frac{l}{4}, \quad \varkappa = \frac{2 - g + \sqrt{g(g-4)}}{2 - g - \sqrt{g(g-4)}}, \quad g = (4\pi r)^2 \left(x_1 \frac{dx_1}{dz}\right)^2_{z=0} \tag{3.12}$$

As can be seen from (3.4), $g(z_1) = 0$ when $z_1 = 0$. Hence, to satisfy the condition $\mu_{z_1 = 0}' = \mu$ we must put $l = 3$. As can be seen from the above μ' is a complex function of $f^2, f_*^2, z_m^2, \alpha_0^2$, and c^2. Since $g_{z_1 = 0} = 0$ and since z_1 determines the distance between the poles, it seems reasonable to represent g in the form of a series in powers of z_1. Assuming that C_k [see (3.4)] themselves depend on z_1, and grouping terms, we have

$$\sum_{k=0}^{\infty} (-1)^k C_k(z_1) z_1^k = \nu_0 + \nu_1 z_1^2 + \nu_2 z_1^4 + \ldots$$

$$\nu_0 = \sum_{k=0}^{\infty} \nu_{0k}, \quad \nu_1 = \sum_{k=2}^{\infty} \nu_{1k}, \quad \nu_2 = \sum_{k=3}^{\infty} \nu_{2k} \tag{3.13}$$

$$\nu_{0n} = -\frac{1}{n(n+2)}\left[\sum_{k=0}^{n-1} \nu_{0k} p_{(n-k)}\right], \quad \nu_{1n} = -\frac{1}{n(n+2)}\left[\sum_{k-2}^{n-1} \nu_{1k} p_{(n-k)} + \beta_2 \nu_{0,(n-2)}\right]$$

$$\nu_{2n} = -\frac{1}{n(n+2)}\left[\sum_{k=3}^{n-1} \nu_{2k} p_{(n-k)} + \beta_2 \nu_{1,(n-2)} + \beta_3 \nu_{0,(n-3)} + \beta_4 \nu_{0,(n-4)}\right]$$

$$\nu_{00} = 1, \quad \nu_{12} = \Omega^2 \alpha_0^2 / 8, \quad \nu_{23} = 2/15\ \Omega^2 u^2$$

$$p_k = -(3k-5)2^{-(k+2)}, \quad \beta_2 = -\Omega^2 \alpha_0^2, \quad \beta_3 = -2\Omega^2 u^2, \quad \beta_4 = \Omega^2 u^2$$

where $\nu_0, \nu_1, \nu_2 \ldots$ do not depend on z_1. Analysis of the expressions for $\nu_{0n}, \nu_{1n}, \nu_{2n}$ and numerical calculations show that beginning with a certain n, ν_{0n} and the numerical coefficients in ν_{1n}, ν_{2n} decrease as n increases no more slowly than the terms of a geometrical progression with a common ratio of 1/2. This enables us, when calculating ν_0, ν_1, ν_2, to confine ourselves to certain terms of the series and to estimate the remainder. We thereby obtain the series

$$g = (4\pi r)^2 z_1^4 [1.7776 + 1.6011\alpha_0^2 \Omega^2 z_1^2 + (0.5942\alpha_0^4 \Omega^4 + 1.2168\Omega^2 u^2) \times z_1^4 + \ldots] \tag{3.14}$$

The procedure described above for obtaining the relationship between the asymptotic solutions can be applied to the more general equation

$$\frac{d^2 y}{dz^2} + \left[\frac{Q_1 z}{z^2 - z_1^2} + a_1 z\right]\frac{dy}{dz} + \left[\frac{Q_2 z^2}{(z^2 - z_1^2)^2} + \frac{Q_3}{z^2 - z_1^2} + a + bz^2\right] y = 0 \tag{3.15}$$

where Q_1, Q_2, Q_3, a_1, a, and b are arbitrary parameters.

4. For Eq. (3.1)

$$\eta = \gamma + i\delta, \quad \gamma = -\frac{\pi}{2}\frac{z_m}{\lambda_*}\frac{f^2}{f_*^2} s, \quad \delta = \frac{\pi}{2}\frac{z_m}{\lambda_*}\left(1 - \frac{f^2}{f_*^2} + \alpha_0^2 \frac{f^2}{f_*^2}\right) \tag{4.1}$$

We will investigate μ'. From the symmetry properties of $C_k(z_1)$ with respect to z_1 described above, and also from (3.4) and (3.12), it follows that $g \geq 0$ for $s = 0$. Suppose $s = 0$ and $0 \leq g \leq 4$. Then [see (3.12)] $|\varkappa| = 1$, $0 \leq \arg\varkappa \leq 2\pi$, and correspondingly $3/4 \leq \mu' \leq 1$, i.e., for the given variation in g the variable under the logarithm sign goes around the branch point of the logarithm and this branch changes into the other. Hence, for $g > 4$ we must take $\ln\varkappa + i2\pi$. Let $s = 0$ and $g \geq 4$. In this case $|\varkappa| > 1$, $\arg \varkappa = 0$, and taking the above into account $\mu' = 1 + i\psi$, where $\psi \geq 0$.

To choose the direction in which to go around the singularities we assume

$$s = 0, \quad 0 \leqslant g \leqslant 4 \qquad (3/4 \leqslant \mu' \leqslant 1)$$

In this case we obtain from (2.6) using Eqs. 8.334.2 of [4]

$$|D|^2 = \frac{1}{4}\frac{e^{-2\pi\delta}}{\cos^2 \pi\mu' + \operatorname{sh}^2 \pi\delta}, \quad |R|^2 = |D|^2 (q_{\pm} 2\cos 2\pi\mu' + e^{2\pi\delta})^2 \tag{4.2}$$

A simple check of these equations shows that for $3/4 \leq \mu' \leq 1$ we have $|R|^2 + |D|^2 \geq 1$ when going around the singularity in the upper half-plane and $|R|^2 + |D|^2 \leq 1$ when going around in the lower half-plane, so that in both cases $|R|^2 + |D|^2 = 1$ only for $\mu' = 3/4$. Hence it follows that a physically correct result is obtained by going around the singularities in the lower half-plane. Thus the correct expressions for R and D in the general case are

$$D = (2\pi)^{-1} e^{i\pi(\eta-1)} \Gamma(1/2 + \mu' - \eta) \Gamma(1/2 - \mu' - \eta), \quad R = De^{-i2\pi(\eta - 3/4)} \tag{4.3}$$

For the case s = 0, g ≥ 4, ($\mu' = 1 + i\psi$), using Eqs. 8.331 and 8.332.2 of [4] we have from (4.3)

$$|D|^2 = \frac{1}{2}\frac{0.25-(\psi-\delta)^2}{0.25+(\psi+\delta)^2}\frac{e^{-2\pi\delta}}{\operatorname{ch}2\pi\psi+\operatorname{ch}2\pi\delta}, \quad |R|^2 = |D|^2 e^{4\pi\delta}$$

If we assume s = 0, and $\eta = 0$ ($f = f_*/\cos\theta_0$), the expressions for R and D further simplify to

$$D = -\frac{1}{2\cos\pi\mu'}, \quad R = -iD \tag{4.5}$$

The quantity $|F|^2 = 1 - |R|^2 - |D|^2$ for s = 0 represents the relative fraction of the energy which is absorbed in the region of the pole. (As can be seen from (4.2), (4.4), and (4.5), $|F|^2 = 0$ only when $\mu' = 3/4$). Since it is in the region of the pole ($\varepsilon' = 0$, s = 0) that conversion of the electromagnetic wave into a plasma wave occurs (see [1]), for a wave of small amplitude the loss of energy in the region of the pole when s = 0 is due to conversion. On the other hand, such a pole occurs if we neglect the spatial dispersion, and consequently, in the limiting case it reflects the effect of the neglected terms. Hence, $|F|^2$ for s = 0 can be regarded as the electromagnetic-plasma wave conversion factor. As can be seen from (1.1), (1.2), (1.3), (3.3), and (3.4), H_x is finite at $z = \pm z_1$, whereas the quantities E_y and E_z become infinite at these points as $(z \mp z_1)^{-1}$.

It follows from the form of the asymptotic solutions and also from (4.3) that the phase shift φ between the reflected and incident waves at the beginning of the layer ($z = -z_m$) is given by

$$\varphi = 2\pi\frac{z_m}{\lambda_*} - 2\delta\ln 2\pi\frac{z_m}{\lambda_*} + \pi\gamma - \frac{\pi}{2} - \arg\Gamma(1/2 + \mu' - \eta)\,\Gamma(1/2 - \mu' - \eta)$$

Hence when using Eq. 8.362.1 of [4] we can obtain the "group delay time" Δt_{**} from the relation

$$\Delta t_{**} = \frac{d\varphi}{d\omega} = -2\frac{d\delta}{d\omega}\ln 2\pi\frac{z_m}{\lambda_*} + \pi\frac{d\gamma}{d\omega} - \operatorname{Im}2\left\{\mu'\frac{d\mu'}{d\omega}\sum_{k=0}^{\infty}\frac{1}{\chi_k} + \left[0{,}577215 + \frac{\sigma}{\chi_0} - \sum_{k=1}^{\infty}\frac{\chi_0 + k\sigma}{k\chi_k}\right]\frac{d\eta}{d\omega}\right\} \tag{4.6}$$

$$\sigma = 1/2 - \eta, \qquad \chi_k = (\sigma + k)^2 - \mu'^2$$

5. As can be seen from the above discussion, a characteristic feature of problems involving a wave equation which has a first- or second-order pole, or, more accurately, a finite number of normal singular points in a finite part of the z-plane, is the nonuniqueness of the results. This nonuniqueness can be eliminated in the general case in the following way. A solution is chosen in the positive (negative) semiaxis z outside the region where all the poles of the equation are studied. A further circuit is made around all these regions in the lower and upper half-planes on the negative (positive) semiaxis z. These paths, generally speaking, are not equivalent. One of the paths is eliminated from physical considerations (for example, it gives $|R| > 1$ or $(|R|^2 + |D|^2) > 1$ etc.). Thus R, D, etc., are uniquely determined by the parameters of the wave equation.

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